Mathematics for Engineers II. lectures

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Differential Equations

This work was supported by the construction EFOP-3.4.3-16-2016-00021. The project was supported by the European Union, co-financed by the European Social Fund.

Examples for differential equations

Newton's second law for a point mass

Consider a particle of mass m subject to net force[a] F. Newton's second law states that the vector acceleration a of the particle is caused by the net force F and is proportional to that force:

$$F = ma, \qquad \text{or} \qquad a = \frac{1}{m}F,$$

where the coefficient of proportionality m is referred to as the (inertial) mass.

[a] By net force, also called the resultant force, one means the vector sum of all forces acting on the particle.

Examples for differential equations

EQUATION OF FREE FALL: Let's denote by $x(t)$ the position of the point mass at time instant t. Then the velocity of the particle is $\dot{x}(t)$ and its acceleration is $\ddot{x}(t)$. Since the acceleration of a freely falling mass is constant (because F is also constant in this case) we have

$$\ddot{x}(t) = -g,$$

where $g \approx 9.81 \frac{m}{s^2}$ is the acceleration due to gravity of Earth.

Examples for differential equations

Integration twice the previous equation we get

$$x(t) = -\frac{gt^2}{2} + c_1 t + c_2,$$

where c_1, c_2 are arbitrary constants.
Assuming zero velocity at the time instant $t = 0$ and starting the free fall from height $\bar{s}$ one can derive the following **initial conditions**:

$$x(0) = \bar{s}, \qquad x'(0) = 0.$$

One can determine now the values of c_1 and c_2 and the solution which fulfils the initial conditions is

$$x(t) = -\frac{gt^2}{2} + \bar{s},$$

which gives the position of a point mass depending on time t falling freely from hight $\bar{s}$.

Examples for differential equations

EQUATION OF HARMONIC OSCILLATION: The next simple example is, when $F = -kx$, $(k > 0)$. The minus sign indicates that the force is restoring. One can think of F as the tension of a linear zero length spring, i.e., of a spring whose relaxed length is zero. Then Newton's equation becomes

$$x''(t) = -\omega^2 x(t),$$

where $\omega^2 = \frac{k}{m}$. The solution will be

$$x(t) = a\cos(\omega t + \varphi),$$

where a is the amplitude and φ is the phase are arbitrary constants which can be determined from the initial data.

A GEOMETRIC EXAMPLE: Assume that the slope of a function at a point $(t, x(t))$ equals to the sum of the coordinates of the point. Then we have the differential equation

$$x'(t) = t + x(t)$$

whose general solution is:

$$ce^t - t - 1.$$

Examples for differential equations

EQUATION OF RADIOACTIVE DECAY: Denote by $x(t)$, $t \geq 0$ the amount of the material which is not decayed at the time instant t. According to experience the amount of the decayed material (during one unit of time) is proportional to the amount of the remaining material:

$$x(t) - x(t+1) = \alpha x(t),$$

where α is a parameter which depends on only the material. Take h as a time unit, then our equation can be written as

$$x(t) - x(t+h) = \alpha(h)x(t).$$

It can be derived that α is monotone decreasing and there is a β such that $\lim_{h\to 0} \frac{\alpha(h)}{h} = \beta > 0$. Taking the limit $h \to \infty$ we have the following ODE

$$x'(t) = -\beta x(t).$$

Its general solution is

$$x(t) = ce^{-\beta t}.$$

Population growth model: Let $x(t)$ be the size of a population at time t. If the relative growth rate of the population per unit time is denoted by $c(t, x(t))$, then

$$x'(t) = c(t, x(t))x(t).$$

In practice the population cannot be arbitrary large. The number N denotes the size of the largest population that can be supported by the system. We assume that the relative population growth rate depends only on x, and it tends to zero as the size of the population approaches N. In particular, we assume that c is given by one of the followings:

$$c(x(t)) = \alpha(N - x(t))^k, \qquad k \in \{0, 1, 2\}.$$

The case $k = 1$ results the **logistic equation**.
At $t = 0$ we have the value $\alpha = c_0(N - x_0)^{-k}$. Substituting this into the equation using the initial conditions $x_0 = 1$ and $N = \beta x_0$ we get

$$x' = c_0 \left(\frac{\beta - x}{\beta - 1}\right)^k, \qquad k \in \{0, 1, 2\}.$$

If $k = 0$ or $k = 1$ the solution is not difficult.

- Galileo Galilei drop down two balls of the same size from the 56 meters hight Leaning Tower of Pisa. The first one is made of metal (100 kilograms), the second one is made of wood (10 kilograms). What happens? Which ball falls down faster? Calculate the time is needed!
- The initial mass of an Iodine isotope was 200g. Determine the Iodine mass after 30 days if the half life of the isotope is 8 days.
- The cooling velocity of a body is proportional the difference between temperature of the environment and the temperature of the body. Let's denote by T the former and by T_k the latter. A body with temperature 100 Celsius in an environment with 0 Celsius cools down 50 Celsius in 20 minutes. What is the temperature of the body after 10 minutes cooling? Use the Heat Equation

$$\frac{\partial T}{\partial t} = -k(T - T_k),$$

where k depends on the material.

Classification of differential equations

A differential equation is an equation containing independent variables, functions, and derivatives of functions. An equation involving derivatives of one variable functions is said to be an **ordinary differential equation ODE** for short.

Examples for ODE

- $x'(t) = t^3 x(t)$,
- $x'(t) = 4t\sqrt{x(t)},\ x(1) = 1$,
- $x'(t) = \log(x(t))$,
- $x'(t) = (1 + x^2)\log t$.

An equation involving partial derivatives of multivariable functions is called **partial differential equation** for short **PDE**.

Examples for PDE

- $\frac{\partial u}{\partial t}(t, x) = 0$,
- $\frac{\partial^2 u}{\partial t^2}(t, x) + \frac{\partial^2 u}{\partial x^2}(t, x) = 0$,
- $\frac{\partial u}{\partial t}(t, x_1, \dots, x_n) - \frac{\partial^2 u}{\partial x_1^2}(t, x_1, \dots, x_n) - \cdots - \frac{\partial^2 u}{\partial x_n^2}(t, x_1, \dots, x_n) = e^t$.

Classification of differential equations

A system of equations involving derivatives or partial derivatives of functions is called a system of ordinary differential equations or a system of partial differential equations respectively.
An ODE can be **linear** or **nonlinear**. For example the general form of the first order, linear equation is

$$\mathbf{x}'(\mathbf{t}) + \mathbf{g}(\mathbf{t})\mathbf{x}(\mathbf{t}) = \mathbf{f}(\mathbf{t}).$$

Examples for linear ODE

- $x'(t) - tx(t) = t^3$,
- $x'(t) + x(t) = e^{-t}$,
- $x'(t) + x(t)\tan t = \sin 2t \;\; t \in]0, \frac{\pi}{2}[$,
- $tx'(t) + 2x(t) = 3t, \;\; x(1) = 0$,
- $x'(t) = x(t) + t$,
- $x''(t) = -tx(t) + t$.

Classification of differential equations

Examples for nonlinear ODE

- $x'(t)x^3(t) = t^3$,
- $x'^2(t) + x(t) = 0$,
- $x'(t) + \tan(x(t))t = \sin 2t \;\; t \in]0, \frac{\pi}{2}[$,
- $x'(t) + x(t) = -\frac{1}{x(t)}$.

If the highest order derivative is n in the equation, then it is said to be an ***n*th order equation**.

Examples for higher order ODE

- $x'' - x' - x = 0$,
- $a_n(t)x^{(n)}(t) + \cdots + a_1(t)x'(t) + a_0(t)x(t) = f(t, x(t), \ldots, x^{(n)}(t))$,
- $y'''(x) - y'(x) = 0, \;\; y(0) = 3, \;\; y'(0) = -1, \;\; y''(0) = 1$,
- $z(s) - z'''(s)z'(s) = z''(s)$.

An nth order ODE can be written as a system of first order ODEs containing n equations.

The concept of ODE

Let $I \subset \mathbb{R}$ be an open, nonempty interval and $F\colon I \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a given function. The equation

$$\text{(IKDE)} \qquad F(t, x, x', \dots, x^{(n)}) = 0$$

is called an n**th order, implicit, ordinary differential equation**. If there exists $f\colon I \times \mathbb{R}^n \to \mathbb{R}$, with the property $F = x^{(n)} - f$, then (**??**) can be written in the form

$$\text{(EKDE)} \qquad x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$

which is called an n**th order, explicit, ordinary differential equation**. If there are functions $a_i\colon I \to \mathbb{R},\ i = 0, \dots, n-1$ such that $f = \sum_{i=0}^{n-1} a_i x^{(i)}$, in detail

$$\text{(LKDE)} \qquad x^{(n)} = a_0(t)x + a_1(t)x' + \dots + a_{n-1}(t)x^{(n-1)} + h(t),$$

then the equation (**??**) is said to be an n**th order, linear ODE**.

Explicit first order ODE

Let $D \subset \mathbb{R}^2$, $f: D \to \mathbb{R}$, then the equation

(EE-KDE) $$x' = f(t, x)$$

is called an **explicit, firs order ODE**.

Solution of (**??**)

Let I be an interval. The function $x\colon I \to \mathbb{R}$ is a **solution of** (**??**), if

- the graph of x is in D, that is to say

$$(t, x(t)) \in D, \qquad t \in I,$$

- x is differentiable,
- x fulfils the equation, that is

$$x'(t) = f(t, x(t)), \qquad t \in I.$$

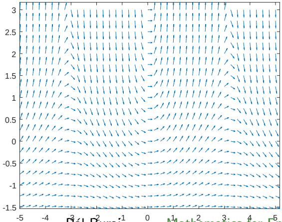
Direction field

Using the previous notations if $t \in I$, then $(t, x(t), f(t, x(t))$ the triple is called a **line element**. The collection of all line elements

$$IM := \{\ (t, x(t), f(t, x(t)) \mid t \in I\ \}$$

is called a **direction field** (more precisely it is the direction field belonging to(**??**)).

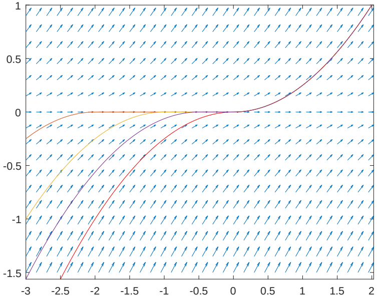
The connection between direction fields and the differential equation can be expressed in geometric terms as follows: A solution x of a differential equation "fits" its direction field, i.e., the slope at each point on the solution curve agrees with the slope of the line element at that point. For example the direction field of equation $x' = e^x \sin(t)$ is:



The previous geometric interpretation suggests that there is a unique solution of the equation at every point. This is not true in general. Lets consider the equation below together with the parametric solution family $(a < 0)$:

$$x' = \sqrt{2|x|}, \qquad x_a(t) = \begin{cases} \frac{t^2}{2} & \text{if } t > 0 \\ 0 & \text{if } a < t \leq 0 \\ \frac{-(t-a)^2}{2} & \text{if } t \leq a. \end{cases}$$

Then this equation has infinitely many solution with the initial condition $x(0) = 0$.



Initial value problem

Using the previous notations the following pair

$$x'(t) = f(t, x(t)), \qquad x \in I, \tag{1}$$

$$x(\xi) = \eta \tag{2}$$

is said to be an **initial value problem**. The second equation is the **initial condition**.

The example on the previous slide shows that the initial value problem is not uniquely solvable in general.

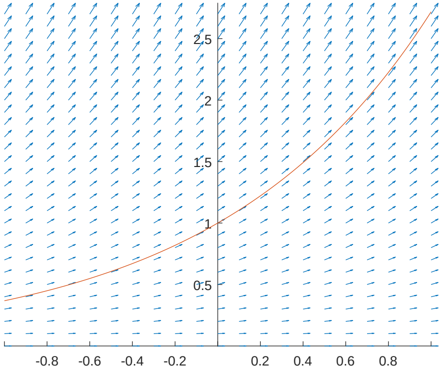
We will prove later, that it is uniquely solvable if we add a simple assumption to the initial value problem. Existence and uniqueness are very important questions considering applications in particular numerical solution of ODE.

Example

Let's consider the following initial value problem:

$$x' = x$$
$$x(0) = 1$$

In the direction field of the equation (in blue) it is drawn (in red) the unique solution of the above mentioned initial value problem.



Elementary solution methods

$x' = f(t)$

If the right hand side does not depend on the unknown function, then we get the solution after a simple integration of the equation.

$$x(t) = \int_{\xi}^{t} f(t)\,\mathrm{dt}$$

This solution fulfils the initial condition $x(\xi) = 0$.

Example

The solutions of the equation

$$x' = t^3 + \cos t$$

has the form

$$x(t) = \frac{1}{4}t^4 + \sin t + C.$$

Determine the solution which fulfils the initial condition $x(1) = 1$.

Elementary solution methods

$x' = g(x)$

Assuming heuristically t can be written as a function of x, rearranging the equation we have

$$\int \frac{\mathrm{dx}}{g(x)} = \int 1 \,\mathrm{dt} = t + C,$$

where C is an arbitrary constant.

Example

$$x' = x,$$

the nowhere zero solutions are

$$\int \frac{dx}{x} = \int 1 \,\mathrm{dt} = t + C \Rightarrow \log(|x|) = t + C \Rightarrow x(t) = Ke^{t},$$

where K is an arbitrary constant.

Separable differential equations

(3) $$x' = f(t)g(x)$$

Heuristic: let's write the equation into the following form

$$\frac{dx}{dt} = f(t)g(x).$$

Rearranging the equation ("separating" the variables), after integration we have

$$\int \frac{\mathrm{dx}}{g(x)} = \int f(t)\,\mathrm{dt}\,.$$

Elementary solution methods

If we apply the previous heuristic for the following initial value problem

$$x' = f(t)g(x), \qquad x(\xi) = \eta \tag{4}$$

we get the equation below:

$$\int_{\eta}^{x} \frac{\mathrm{da}}{g(a)} = \int_{\xi}^{t} f(b)\,\mathrm{db} \tag{5}$$

Theorem

Assume that

- f *is continuous on the interval* I_t*,*
- g *is continuous on the interval* I_x*,*
- η *is an interior point of* I_x *and* $g(\eta) \neq 0$*.*

Then there exists a neighbourhood of ξ *(it can be a one-sided neighbourhood if* ξ *is a boundary point) in which the initial value problem* (**??**) *has a unique solution* $x(t)$*. It can be obtained from equation* (**??**) *by solving for* x*.*

Exercises

- $x' = \frac{1}{t+a}, \quad a \in \mathbb{R},$
- $z' = \frac{1}{2+3t^2},$
- $u' = cos(t),$
- $y' = x,\ y(0) = 1,$
- $x' = -x,\ x(1) = -1,$
- $y' = xe^x,$
- $x' = te^x,$
- $(t^2 - 1)x' + 2tx^2 = 0.$

Linear differential equations

Let $g, h\colon I \to \mathbb{R}$ be continuous functions, then the equation

$$x' + g(t)x = h(t), \qquad t \in I \tag{6}$$

is called a **linear differential equation**. If $h \equiv 0$ then the equation is **homogeneous**, otherwise it is **inhomogeneous**.

Solution of the homogeneous equation

This is a special separable equation, its solution is

$$x(t; C) = Ce^{-G(t)}, \quad \text{where} \quad G(t) = \int_{\xi}^{t} g(a)\,\mathrm{da},$$

where $\xi \in I$ is fixed. The solution which fulfil the initial condition $x(\xi) = \eta$ can be written as

$$x(t) = \eta e^{-G(t)}.$$

Linear differential equations

Solution of the inhomogeneous equation

We are looking for the solution in a similar form, $x(t; C) = Ce^{-G(t)}$, as in the homogeneous case. However, C is not constant here, but a function of t.

Example

Lets consider the inhomogeneous equation $x' - x = t$.
Homogeneous part: $x' = x \Rightarrow x = ce^t$. **Method of variation of constants:** $c \rightsquigarrow c(t)$, substitution, $x(t) = c(t)e^t$ and $x'(t) = c'(t)e^t + c(t)e^t$. Substituting back into the original equation:

$$\underbrace{c'(t)e^t + c(t)e^t}_{x'} - \underbrace{c(t)e^t}_{x} = t \Rightarrow c' = te^{-t} \Rightarrow c(t) = e^{-t}(1-t) + K,$$

where K is an arbitrary constant.
Using this, the solution of the original equation is:

$$x = 1 - t + Ke^t, \qquad \text{where } K \text{ is an arbitrary constant.}$$

Linear differential equations

Exercises

- $x' + 2tx = 0$,
- $tx' - x = 0$,
- $x' - \frac{x}{t} = t^2 + 3t - 2$,
- $(t-2)x' - x = 2(t-2)^3$.

Bernoulli equation

Jacob Bernoulli (1654-1705) Swiss mathematician discovered the following nonlinear differential equation, which is a special case of the logistic equation.

$$x' + g(t)x + h(t)x^{\alpha} = 0, \qquad \alpha \neq 1.$$

Multiply the equation by $(1-\alpha)x^{-\alpha}$ then the function $y = x^{1-\alpha}$ is a solution of the linear equation

$$y' + (1-\alpha)g(t)y + (1-\alpha)h(t) = 0.$$

Nonlinear equations

Example

$x' - x - tx^5 = 0$ multiply by $-4x^{-5}$ we have $-4x^{-5}x' + x^{-4} + 4t = 0.$

After the substitutions $y = x^{-4}, \quad y' = -4x^{-5}x'$ we have to solve the linear ODE

$$y' + 4y + 4t = 0,$$

its solution is

$$x^{-4} = y = ce^{-4t} - t + \frac{1}{4}.$$

Exercises

- $3x' + x = (1 - 2t)x^4,$
- $x' + x = x^2(\cos t - \sin t).$

Riccati equation

The nonlinear equation below was discovered by Jacopo Riccati (1676-1754) from Venice.

$$x' = q_0(t) + q_1(t)x + q_2(t)x^2$$

Except in special instances, the solutions cannot be given in closed form. However, if one solution is known, then the remaining solutions can be explicitly calculated.

Let φ be a given solution, then $u := x - \varphi$ is a solution of the equation

$$\underbrace{(x-\varphi)'}_{u'} = q_1(t)\underbrace{(x-\varphi)}_{u} + q_2(t)(y^2 - \varphi^2) = q_1(t)u + q_2(t)\underbrace{(x-\varphi)}_{=u}\underbrace{(x+\varphi)}_{=u+2\varphi},$$

where x is an arbitrary unknown solution of the original equation. From this we have the Bernoulli equation for u

$$x' = \big(q_1(t) + 2q_2(t)\varphi\big)u + q_2(t)u^2.$$

Nonlinear equations

Exercises

Solve the Riccati equations using the given particular solutions:

- $y'(x) + \frac{1}{x}y(x) + y^2(x) = \frac{4}{x^2}, \quad y_p(x) = \frac{2}{x}$,
- $y'(x) + \frac{1}{3}y^2(x) + \frac{2}{3}\frac{1}{x^2} = 0, \quad y_p(x) = \frac{1}{x}$,
- $y'(x) + 2y(x)e^x - y^2(x) = e^{2x} + e^x, \quad y_p(x) = e^x$,
- $y'(x) - \frac{y(x)}{x} = y^2(x) + \frac{1}{x^2}, \quad y(x)_p = \frac{c}{x}$.

System of equations and higher order equations

Definition

The system of equations

$$\begin{aligned} x_1' &= f_1(t, x_1, \dots, x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, \dots, x_n) \end{aligned}$$

is called a **first order, explicit, differential equation**, where $f_i\colon D \subset \mathbb{R}^{n+1} \to \mathbb{R},\ i = 1, \dots, n$ are given functions.
The vector function $x^T = (x_1, \dots, x_n)$ is a **solution** of the system if $(t, x) \in D$ and the components of x fulfils the system.

Using the previously introduced vector notation, the system can be written in the following closed form

$$x' = f(t, x),$$

where $f^T = (f_1, \dots, f_n)$.

Differentiation and integration of vector functions is understood coordinate-wise. Using the vector notation the problem

$$x' = f(t, x), \quad x(\xi) = \eta, \qquad \xi \in \mathbb{R},\ \eta \in \mathbb{R}^n$$

is called an **initial value problem with respect to the first order, explicit, differential equation** $x' = f(x, t)$. One can pose an analogous existence and uniqueness theorem for this problem as in the one-variable case.

Definition

The equation

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)})$$

is called an *n***th order, explicit, ordinary differential equation**.

System of equations and higher order equations

The nth order equation can be transformed into a first order, explicit system of ODE in the following way:

$$x_1' = x_2,\ x_2' = x_3, \ldots, x_{n-1}' = x_n,\ x_n' = f(t, x_1, \ldots, x_{n-1}).$$

The nth order equation and the system of equations are equivalent in the following sense: If x is a solution of the nth order equation, then the vector function $x = (x_1, \ldots, x_n)$ is a solution of the system. Conversely, if x is a differentiable solution of the system, and one sets $x =: x_1$, then x is n-times differentiable, $x = (x_1, \ldots, x_n) = (x, x', \ldots, x^{(n-1)})$ is a solution of the system.

This relation will be important later in the numerical solution of higher order equations!

Second order, linear ODE

Linear equation of order 2 with constant coefficients

$$ax'' + bx' + cx = 0,$$

where $a, b, c \in \mathbb{R}$ are given constants. The quadratic equation

$$a\lambda^2 + b\lambda + c = 0$$

is said to be the **characteristic equation of the linear equation of second order with constant coefficients**.

Definition

Two functions are called **linearly independent** (on the intersection of their domains) if the zero function can be combined linearly from them only in a trivial way, that is to say, φ, ψ linearly independent if $\lambda\varphi + \mu\psi = 0$ if and only if $\lambda = \mu = 0$.

Example

Functions $\varphi(t) = e^{\alpha t}$ and $\psi(t) = e^{\beta t}$ $\alpha \neq \beta$ are linearly independent.

Solution of linear equation of second order with constant coefficients

If the the linearly independent functions φ_1 and φ_2 are the solutions of the homogeneous equation, then all the solutions can be written as their linear combination:

$$c_1\varphi_1 + c_2\varphi_2,$$

where c_1, c_2 are arbitrary constants.

Second order, linear ODE

Let λ_1, λ_2 be the roots of the characteristic equation. Then the following cases are possible:

- We have two different real roots, then $\varphi_1(t) = e^{\lambda_1 t}$ and $\varphi_2(t) = e^{\lambda_2 t}$ are linearly independent solutions.
- We have one real double root $\lambda_1 = \lambda_2 = \lambda$. Then $\varphi_1(t) = e^{\lambda t}$ and $\varphi_2(t) = te^{\lambda t}$ are linearly independent solutions.
- We have two complex roots $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. Then $\varphi_1(t) = e^{\alpha t}\cos(\beta t)$ and $\varphi_2(t) = e^{\alpha t}\sin(\beta t)$ are linearly independent solutions.

Example

Let's consider the equation $x'' - x' - 6x = 0$. The roots of the characteristic equation are $\lambda_1 = 3$ and $\lambda_2 = -2$, so the general solution of the differential eqaution is:

$$x(t) = c_1 e^{3t} + c_2 e^{-2t}$$

Second order, linear ODE

Exercises

Solve the following differential equations!

- $x'' - 8x' + 16 = 0$,
- $4x'' + 4x' + 37x = 0$,
- In case of undamped oscillation the acceleration is proportional to the movement, but their directions are opposite. Determine the movement if the velocity is $v_0 = c_2\omega$ and the movement is 0 at $t = 0$.
  The equation of undamped oscillation:
  $$\ddot{y} = \omega^2 y,$$
  where ω is the angular velocity.
- Solve the equation of damped oscillation using the previous initial data
  $$m\ddot{y} = -\omega^2 my - 2s\dot{y},$$
  where m is the mass, and s is the damping factor.

Existence and uniqueness theorems

Theorem (Picard-Lindelöf existence an uniqueness theorem)

Assume that there is a constant $L > 0$*, such that*

$$|f(t,x) - f(t,\bar{x})| \leq L|x - \bar{x}|, \qquad \textit{(Lipschitz condition)}$$

for all fixed $t \in [\xi, \xi + a]$*.* f *is continuous on* $[\xi, \xi + a] \times \mathbb{R}$*. Then the initial value problem* (**??**)-(**??**) *has a unique solution*

$$x\colon [\xi, \xi + a] \to \mathbb{R}$$

on the interval $[\xi, \xi + a]$*.*

Theorem (Peano existence theorem)

If f *is continuous on an open set of the plane which contains the point* (ξ, η)*, then the initial value problem* (**??**)-(**??**) *has at least one solution goes through* (ξ, η) *which is extendible to the boundary of the set.*

Exercises

- Can be applied or not the previous theorems for the following problems?
  - $x' = \sqrt{|x|},\ x(0) = 0,$
  - $y' = y\log(y),\ y(1) = 1.$
- Rephrase the following initial value problems as integral equations!
  - $x' = t - x^2,\ x(0) = 0,$
  - $y' = y^2 - 3x^2 - 1,\ y(0) = 1,$
  - $y' = y + e^y,\ y(0) = 1.$